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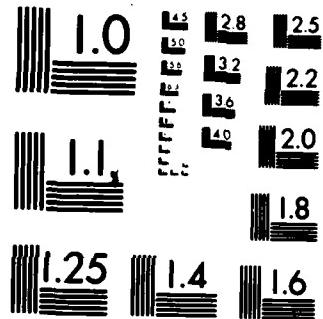
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EXISTENCE FOR A PROBLEM IN GROUND  
FREEZING

E. DiBenedetto and C. M. Elliott

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EXISTENCE FOR A PROBLEM IN GROUND FREEZING

E. DiBenedetto<sup>\*</sup>,<sup>1</sup> and C. M. Elliott<sup>2</sup>

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ABSTRACT

A system of two elliptic p.d.e.'s, which model the conduction-convection problem in a porous medium with change of phase is considered.

The first equation describes the heat conduction and holds in a fixed domain. The second takes into account the convective motions and holds in the unknown melted part of the region.

The existence of a locally regular weak solution is proved by making use of various compactness arguments.

AMS (MOS) Subject Classification: 35J65, 35Q99, 76R05, 76S05, 76T05

Key Words: Darcy's law, convection, free boundary

Work Unit Number 1 (Applied Analysis)

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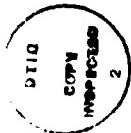
### SIGNIFICANCE AND EXPLANATION

Consider the problem of artificial freezing or thawing of ground water. The problem has important engineering applications such as ground stabilization at construction sites and the prevention of water leakage into tunnels or shafts.

In the melted part of the ground the convective motions forced by temperature gradients affect significantly the free boundary of separation between the melted and solid regions.

As a mathematical model of this situation we study a system of two elliptic equations, one describing heat conduction and the other the velocity flow of convection. We demonstrate that the problem has a solution in a suitable weak sense. Regularity properties of the solution are also derived.

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EXISTENCE FOR A PROBLEM IN GROUND FREEZING

E. DiBenedetto<sup>\*</sup>,<sup>1</sup> and C. M. Elliott<sup>2</sup>

1. INTRODUCTION

There exists laboratory and theoretical evidence to suggest that convection can play an important role in heat transfer with change of phase in a porous medium (see [6, 7, 8, 1]). We mention specifically the artificial freezing or thawing of groundwater in view of the important engineering applications such as ground stabilization at construction sites and the prevention of water leakage into tunnels or shafts (see [11, 12]). In certain circumstances, the moving boundary problem considered by the above authors has a nontrivial steady state. This leads to the study of a free boundary problem for a system of two elliptic partial differential equations.

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  lying between two closed curves  $\partial_+ \Omega$  and  $\partial_- \Omega$ . The domain  $\Omega$  is the union of two sets  $\Omega^-$  and  $\Omega^+$  which are respectively the frozen and unfrozen regions with the phase change temperature being zero at the unknown free boundary  $\Gamma$  separating the sets. Figure 1 can be interpreted as a vertical cross-section of a portion  $\Omega$  of frozen ground, where a horizontal pipe of cross-section  $\partial_+ \Omega$  carries warm oil. Then around the pipe there is an unfrozen region  $\Omega^+$ . If  $\Omega^+$  and  $\Omega^-$  are interchanged then one could regard Figure 1 as depicting the artificial freezing of a horizontal layer of ground by the insertion of a vertical freeze pipe. In this case the experiments of Privik and Comini [6] and the numerical work of Goldstein and Reid [7] and Barrett and Elliott [1], show that the flow of groundwater can significantly affect the extent of the frozen region.

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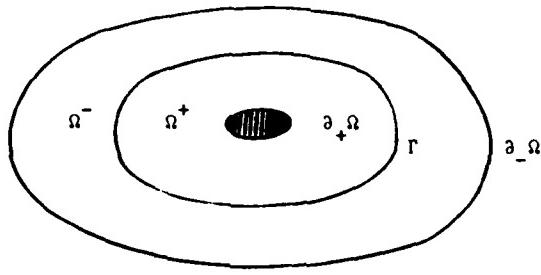


Figure 1

Let  $\hat{u}, \hat{v}$  and  $\hat{p}$  denote, respectively, dimensionless temperature, velocity and pressure. The model equations are then the steady state convection-conduction equation, the continuity equation and Darcy's law which links the pressure to the velocity [1,7]. The problem can be formulated as follows.

Find  $\{\hat{u}, \hat{v}, \hat{p}, \Gamma\}$  such that

$$(1.1) \quad -\Delta \hat{u} + \lambda \hat{v} \cdot \nabla \hat{u} = f_0 + \sum_{i=1}^2 \frac{\partial}{\partial x_i} f_i \quad \text{in } \Omega$$

$$(1.2) \quad \Omega^+ \equiv \{x \in \Omega | u(x) > 0\}, \quad \Omega^- \equiv \{x \in \Omega | u(x) < 0\}$$

$$\Omega = \Omega^+ \cup \Gamma \cup \Omega^-, \quad \partial \Omega \equiv \partial_+ \Omega \cup \partial_- \Omega$$

$$(1.3) \quad \hat{v} = -k_0 \{\nabla \hat{p} + \hat{g}(\hat{u}, x)\} \quad \text{in } \Omega^+$$

$$(1.4) \quad \operatorname{div} \hat{v} = 0 \quad \text{in } \Omega^+, \quad \hat{v} \equiv 0 \quad \text{in } \Omega^-$$

$$(1.5) \quad \hat{v} \cdot \hat{N}_{\partial_+ \Omega} = 0$$

$$(1.6) \quad u = \phi \quad \text{on } \partial \Omega, \quad \phi > 0 \quad \text{on } \partial_+ \Omega, \quad \phi < 0 \quad \text{on } \partial_- \Omega.$$

Here  $\lambda > 0$  is a prescribed physical constant,  $f_0 + \sum_{i=1}^2 \frac{\partial}{\partial x_i} f_i$  represents a known distribution of heat sources (or sinks) in  $\Omega$ ,  $k_0$  is the coefficient of permeability and  $\hat{g}$  is a buoyancy force.

Remarks: (i) If  $f_i$ ,  $i = 1, 2$  is the Heaviside function in the  $x_i$  direction then on the right hand side of (1.1) we allow sources (or sinks), like Dirac masses.

(ii) The problem has been formulated having in mind the case when thawing occurs and  $\Omega$  is a vertical layer. This is then a problem in free convection where the flow is driven by the vertical temperature gradient. In the case of artificial freezing in a horizontal layer the effect of a prescribed flow on the extent of the frozen region is of interest and then (1.5) may be replaced by the prescription of a non-zero normal velocity or pressure.

We observe that the incompressibility condition (1.4) and the Darcy law (1.3) imply that the pressure solves the elliptic equation

$$(1.7) \quad \operatorname{div} k_0 [\nabla p + \overset{\rightarrow}{g}(u, x)] = 0 \quad \text{in } \Omega^+.$$

The aim of this paper is to prove an existence theorem for (1.1)-(1.6) in a suitable weak sense. The difficulty here is represented by the fact that (1.3) and (1.7) are conditions which hold on the unknown region  $\Omega^+$ . They might be interpreted in the sense of distributions over  $\Omega^+$  if  $\Omega^+$  is open. This latter information will be implied by the local Hölder continuity of the temperature. The main points of the proof are the introduction of a penalized problem (in the spirit of [3,4]) and a careful use of the information (1.4) and the fact that the space dimension is two, in order to prove that  $u$  is Hölder continuous. We now come to a precise formulation of our results.

Definition. By a weak solution of (1.1)-(1.7) we mean a triple  $\{u, \overset{\rightarrow}{v}, p\}$  such that

- (i)  $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \cap C^0(\bar{\Omega})$ ,  $u = 0$  on  $\partial\Omega$
- (ii)  $\overset{\rightarrow}{v} \in [L^2(\Omega)]^2$  and  $\overset{\rightarrow}{v} = 0$  a.e. on  $\{u < 0\}$ ,  $\overset{\rightarrow}{v} \cdot \overset{\rightarrow}{N}_{\partial\Omega^+} = 0$
- (iii)  $p \in W^{1,2}(\{u > 0\})$ ,

satisfying (the summation notation is used)

$$(1.8) \quad \int_{\Omega} \{\nabla u - \lambda \overset{\rightarrow}{v} u\} \nabla v dx = \int_{\Omega} f_0 n dx - \int_{\Omega} f_i n_{x_i} dx$$

$$v \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$$

$$(1.9) \quad \int_{\{u>0\}} [\nabla p + \overset{\rightarrow}{g}(u, x)] \cdot \nabla v dx = 0$$

$$v \in W^{1,2}(\Omega) \text{ such that } \sup v \subset \{u > 0\},$$

$$(1.10) \quad \int_{[u>0]} (\vec{v} - k_0 [\nabla p + \vec{g}(u, x)]) \cdot \vec{n} dx = 0$$

$\vec{v} \in [L^2(\Omega)]^2$  such that  $\text{supp } \vec{v} \subset [u > 0]$ .

Remarks: (i) Since we require  $u \in C^\alpha(\bar{\Omega})$ , the set  $[u > 0]$  is open in the relative topology of  $\Omega$  and hence (1.9)-(1.10) are meaningful.

(ii) (1.9)-(1.10) imply that  $\text{div } \vec{v} = 0$  in  $D'([u > 0])$ .

The integrals in (1.8)-(1.10) are well defined, modulo basic assumptions listed below.

[A<sub>0</sub>]  $\partial\Omega$  satisfies the cone condition (see [10]).

[A<sub>1</sub>]  $f_i \in L^{2+\kappa}(\Omega)$ ,  $\kappa > 0$ ,  $i = 0, 1, 2$

[A<sub>2</sub>]  $\phi \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ ,  $\phi > 0$  on  $\partial_+ \Omega$  and  $\phi < 0$  on  $\partial_- \Omega$

[A<sub>3</sub>]  $\vec{g} : \mathbb{R}^3 \times \mathbb{R}^2$  is continuous and

$$|\vec{g}(u, x)| \leq C|u|^t + g_0(x),$$

where  $C$  and  $t$  are given nonnegative constants with  $t \in [0, 1]$ , and

$$g_0(x) \in L_2(\Omega).$$

Theorem. Let [A<sub>0</sub>]-[A<sub>3</sub>] hold. Then problem (1.1)-(1.7) has a weak solution.

Remark. Let  $t = 1$  in [A<sub>3</sub>]. The existence of a weak solution in this case follows from a remark in section 5 provided the data is sufficiently small.

The paper is organized as follows. In section 2 a family of penalized (or smoothed) problems is introduced. Basic results for this family are stated in section 2 and proved in sections 4 and 5. The theorem is demonstrated in section 3. Standard notation for function spaces is employed throughout the paper and  $\|\cdot\|_{p, \Omega}$  is used to denote the  $L_p^p(\Omega)$  norm. The measure of a set  $A$  in  $\mathbb{R}^N$  is denoted by  $|A|$ .

2. CONSTRUCTION OF THE SOLUTION

For  $\epsilon > 0$  consider the Lipschitz continuous function  $s \mapsto k_\epsilon(s)$  defined by

$$(2.1) \quad k_\epsilon(s) = \begin{cases} k_0 & s > 0 \\ k_0 + \epsilon^{-1}(k_0 - \epsilon)s & s \in [-\epsilon, 0] \\ \epsilon & s < -\epsilon \end{cases}$$

and the problem of finding functions  $u_\epsilon, \vec{v}_\epsilon, p_\epsilon$  satisfying

$$(2.2) \quad \begin{aligned} u_\epsilon &\in W^{1,2}(\Omega), \quad u_\epsilon = \phi \text{ on } \partial\Omega \\ \vec{v}_\epsilon &\in [L^2(\Omega)]^2, \quad \operatorname{div} \vec{v}_\epsilon = 0 \text{ in } D'(\Omega) \\ p_\epsilon &\in W^{1,2}(\Omega), \end{aligned}$$

$$(2.3) \quad \int_{\Omega} (\nabla u_\epsilon \cdot \nabla v - \lambda \vec{v}_\epsilon \cdot \nabla u_\epsilon) dx = \int_{\Omega} f_0 v dx - \int_{\Omega} f_1 v_i dx, \quad \forall v \in \dot{W}^{1,2}(\Omega) \cap L^\infty(\Omega),$$

$$(2.4) \quad \int_{\Omega} k_\epsilon(u_\epsilon) [\nabla p_\epsilon + \vec{g}(u_\epsilon, x)] \cdot \nabla v dx = 0, \quad \forall v \in W^{1,2}(\Omega)$$

and

$$(2.5) \quad \int_{\Omega} (\vec{v}_\epsilon - k_\epsilon(u_\epsilon) [\nabla p_\epsilon + \vec{g}(u_\epsilon, x)]) \cdot \nabla v dx = 0, \quad \forall v \in [L^2(\Omega)]^2.$$

This smoothing of the problem (1.1)-(1.2) has a sound physical basis. In reality it may indeed be the case that the coefficient of permeability takes a tiny non-zero value in the frozen region (see [6]). Hence the study of (2.1)-(2.5) is of interest in its own right.

The proof of the theorem rests upon the following propositions

Proposition 1.

For all  $\epsilon > 0$ , problem (2.2)-(2.5) has a solution. Moreover,  $\forall \epsilon > 0$

$$(2.6) \quad \|u_\epsilon\|_{\infty, \Omega} \leq G_0$$

$$(2.7) \quad \|u_\epsilon\|_{2, \Omega} + \|\nabla u_\epsilon\|_{2, \Omega} \leq G_1$$

$$(2.8) \quad \|\sqrt{k_\epsilon(u)} \cdot \nabla p_\epsilon\|_{2, \Omega} \leq G_2$$

$$(2.9) \quad \|v_\epsilon\|_{2,\Omega} < G_3$$

For constants  $G_j$  depending upon the data in assumptions  $[A_1]-[A_3]$  and independent of  $\epsilon$ .

Proposition 2.

For every compact set  $K \subset \Omega$  there exists constants  $\gamma(K)$ ,  $\alpha \in (0,1)$  depending only upon the data and independent of  $\epsilon$  such that

$$(2.10) \quad |u_\epsilon(x) - u_\epsilon(y)| \leq \gamma(K) |x - y|^\alpha, \quad \forall (x,y) \in K, \quad \forall \epsilon > 0.$$

Remark

If the boundary datum  $\phi \in C^{\bar{\alpha}}$ , then the equi-Hölder continuity of the net  $\{u_\epsilon\}$  carries up to  $\bar{\Omega}$ , with constants  $\gamma$  and  $\alpha$  depending upon the data and  $\bar{\alpha}$ .

The proof of propositions 1 and 2 are postponed to sections 4 and 5.

3. PROOF OF THE THEOREM

Let us assume Propositions 1,2 for the moment and let us conclude the proof of existence by using these facts.

Because of (2.6)-(2.10) subnets out of  $\{\hat{v}_\epsilon\}$  and  $\{u_\epsilon\}$  can be selected (and relabeled with  $\epsilon$ ) such that

$$\begin{aligned}\hat{v}_\epsilon &\rightarrow \hat{v} \text{ weakly in } [L^2(\Omega)]^2 \\ u_\epsilon &\rightarrow u \text{ weakly in } W^{1,2}(\Omega) \\ u_\epsilon &\rightarrow u \text{ uniformly on compact sets of } \Omega.\end{aligned}$$

Letting  $\epsilon \rightarrow 0$  in (2.3) proves (1.8). From (2.4)-(2.5) we have

$$\int_{\Omega} \hat{v}_\epsilon \cdot \nabla \eta \, dx = 0 \quad \forall \eta \in W^{1,2}(\Omega)$$

and hence

$$\operatorname{div} \hat{v} = 0 \text{ in } D'(\Omega).$$

Since  $\{u_\epsilon\}$  are equibounded and equi-Hölder continuous in  $\Omega$ , the uniform limit  $u$  is Hölder continuous in  $\Omega$ , and therefore the sets  $\{u > 0\}$  and  $\{u < 0\}$  are open in the relative topology of  $\Omega$ .

Let  $K$  be compact and contained in  $\{u < 0\}$ . There exists  $\sigma > 0$  and  $\epsilon_0 < \sigma/4$  such that

$$\begin{aligned}u(x) &< -\sigma, \quad \forall x \in K \\ u_\epsilon(x) &< -\sigma/2, \quad \forall x \in K, \quad \forall \epsilon < \epsilon_0 \\ k_\epsilon(u_\epsilon) &= \epsilon, \quad \forall x \in K, \quad \forall \epsilon < \epsilon_0.\end{aligned}$$

By (2.5)

$$\hat{v}_\epsilon = -k_\epsilon(u_\epsilon)[\nabla p_\epsilon + \hat{g}(u_\epsilon, x)] \text{ a.e. in } \Omega$$

and therefore

$$\begin{aligned}\int_K |\hat{v}_\epsilon|^2 \, dx &< 2 \int_K k_\epsilon^2(u_\epsilon) |\nabla p_\epsilon|^2 \, dx + 2 \int_K k_\epsilon^2(u_\epsilon) |\hat{g}|^2 \, dx \\ &< 2 \epsilon \overline{k_\epsilon(u_\epsilon)} \|\nabla p_\epsilon\|_{2,\Omega}^2 + 2 \epsilon^2 \|\hat{g}(u_\epsilon, x)\|_{2,\Omega}^2\end{aligned}$$

Letting  $\epsilon \rightarrow 0$  by the weak lower semicontinuity of the norm we obtain  $\hat{v} = 0$  a.e. in  $K$ , and since  $K$  is an arbitrary compact set in  $[u < 0]$  we have

$$\hat{v} = 0 \text{ a.e. in } [u < 0].$$

It remains to prove (1.9)-(1.10). Let  $K \subset [u > 0]$  be compact. There exist  $\sigma > 0$ ,  $\epsilon_0 > 0$  such that

$$\begin{aligned} u(x) &> \sigma, & \forall x \in K \\ u_\epsilon(x) &> \sigma/2, & \forall x \in K, \forall \epsilon < \epsilon_0 \\ k_\epsilon(u_\epsilon(x)) &= k_0, & \forall x \in K, \forall \epsilon < \epsilon_0. \end{aligned}$$

By (2.8) and the definition of  $k_\epsilon(u_\epsilon)$

$$\|\nabla p_\epsilon\|_{2,K} \leq C_2 / \sqrt{k_0}.$$

The vectors  $\{\nabla p_\epsilon\}$  are therefore weakly compact in  $[L^2(K)]^2$  and for a subnet, relabeled with  $\epsilon_K$

$$\nabla p_{\epsilon_K} \rightarrow \hat{\xi}_K \text{ weakly in } [L^2(K)]^2.$$

Let  $\{K_n\}$  be a family of compact subsets of  $[u > 0]$  such that

$$K_n \subset [u > 0], \quad K_n \subset K_{n+1}, \quad \bigcup_{n \geq 1} K_n = [u > 0].$$

Then by a diagonal process a subnet out of  $\{\nabla p_\epsilon\}$  can be selected (and relabeled with  $\epsilon$ ) such that

$$\nabla p_\epsilon \rightarrow \hat{\xi} \text{ weakly in } [L^2([u > 0])]^2$$

Next we want to identify  $\hat{\xi}$  as the gradient of a function  $p \in W^{1,2}([u > 0])$ .

Let  $A$  be an open set in  $\mathbb{R}^N$ ,  $N > 2$  and let  $D_\sigma(A)$  be the space of all  $C_0^\infty$  vector fields  $\hat{v}$  in  $A$  which are solenoidal ( $\operatorname{div} \hat{v} = 0$ ). We set

$$J_\sigma(A) = \{\text{the } [L^2(A)]^N \text{ closure of } D_\sigma(A)\}.$$

Let also  $G(A)$  denote the space of those square integrable vector fields in  $[L^2(A)]^N$  obtained as gradients of functions in  $W^{1,2}(A)$ .

By Weyl's lemma, [9], the orthogonal decomposition,

$$[L^2(\Omega)]^N = J_\sigma(\Omega) \oplus G(\Omega) ,$$

holds.

We must show that  $\xi \in G([u > 0])$ , or equivalently that

$$(3.1) \quad \int_{[u>0]} \xi \cdot \hat{v} \, dx = 0, \quad \forall \hat{v} \in J_\sigma([u > 0])$$

Since  $D_\sigma([u > 0])$  is dense in  $J_\sigma([u > 0])$  it will suffice to prove (3.1) for all  $\hat{v} \in D_\sigma([u > 0])$ .

Now  $\forall \epsilon > 0 \quad p_\epsilon \in W^{1,2}([u > 0])$ , therefore for all  $\hat{v} \in D_\sigma([u > 0])$  we have

$$\int_{[u>0]} \nabla p_\epsilon \cdot \hat{v} \, dx = \int_{[u>0]} p_\epsilon \operatorname{div} \hat{v} \, dx = 0 .$$

Letting  $\epsilon \rightarrow 0$  (3.1) follows. We will set

$$\xi = \nabla p, \quad p \in W^{1,2}([u > 0]) .$$

Consider now (2.4) and (2.5) where  $n$  and  $\hat{n}$  are supported on some compact set  $K \subset [u > 0]$ . By an argument similar to the one above, if  $\epsilon$  is sufficiently small we must have  $k_\epsilon(u_\epsilon) = k_0$ . Letting  $\epsilon \rightarrow 0$  we recover (1.9)  $\forall n \in W^{1,2}(\Omega)$   $\operatorname{supp} n \subset K$ , and (1.10)  $\forall \hat{n} \in [L^2(\Omega)]^2$   $\operatorname{supp} \hat{n} \subset K$ . Since  $K \subset [u > 0]$  is arbitrary (1.9) and (1.10) follow.

#### 4. A PRIORI ESTIMATES

It is the purpose of this section to prove the following two lemmas.

##### Lemma 1

Let  $\vec{v} \in [L^2(\Omega)]^2$ ,  $\operatorname{div} \vec{v} = 0$  in  $D'(\Omega)$  and  $|\vec{v}|_{2,\Omega} < v$  for a given positive constant  $v$ . If  $[A_0]-[A_2]$  hold then the problem

$$(4.1) \quad \begin{cases} \int_{\Omega} (\nabla u - \lambda \vec{v} u) \nabla v_n dx = \int_{\Omega} f_0^n dx - \int_{\Omega} f_1^n x_1 dx & v_n \in \overset{\circ}{W}{}^{1,2}(\Omega) \subset L^{\infty}(\Omega) \\ |u|_{\partial\Omega} = 0 \end{cases}$$

has a unique solution  $u \in W^{1,2}(\Omega)$ . Also the following a priori estimates hold:

$$(4.2) \quad \|u\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2 \leq Y_0(1+v^2)$$

$$(4.3) \quad \|u\|_{\infty,\Omega} \leq Y_1(1+v)$$

and  $u$  is Hölder continuous on compact subsets of  $\Omega$  with the Hölder exponent  $\alpha$  and Hölder constant  $Y$  depending on  $[A_0]-[A_2]$  and  $v$ .

##### Lemma 2

Let  $u \in L^{\infty}(\Omega)$  be arbitrary and  $\epsilon$  a given positive constant. If  $[A_0, A_3]$  hold then the problem of finding  $p = p(u, \epsilon) \in W^{1,2}(\Omega)$  such that

$$(4.4) \quad \int_{\Omega} k_{\epsilon}(u)[\nabla p + \vec{g}(u, x)] \cdot \nabla v_n dx = 0 \quad v_n \in W^{1,2}(\Omega)$$

has a solution which belongs to a unique class  $[p] \in W^{1,2}(\Omega)/\mathbb{R}$ . The vector field  $\vec{v}(u, \epsilon) = k_{\epsilon}(u)[\nabla p(u, \epsilon) + \vec{g}(u, x)]$  is unique and the following estimates hold:

$$(4.5) \quad \sqrt{k_{\epsilon}(u)} \|\nabla p(u, \epsilon)\|_{2,\Omega} \leq k_0^{1/2} (c|\Omega|)^{1/2} \|u\|_{\infty,\Omega}^t + \|g_0\|_{2,\Omega}$$

$$(4.6) \quad \|\vec{v}(u, \epsilon)\|_{2,\Omega} \leq 2k_0 (c|\Omega|)^{1/2} \|u\|_{\infty,\Omega}^t + \|g_0\|_{2,\Omega},$$

where  $C$ ,  $t$  and  $g_0$  are introduced in  $[A_3]$ . Finally if  $u_1, u_2 \in L^{\infty}(\Omega)$  then

$$(4.7) \quad \|\nabla p(u_1, \epsilon) - \nabla p(u_2, \epsilon)\|_{2,\Omega} \leq \hat{G}_1 (\|u_1 - u_2\|_{\infty, \Omega} + \|g(u_1, x) - g(u_2, x)\|_{\infty, \Omega}) \\ + (\|g_0\|_{2,\Omega} + \|u_1\|_{\infty, \Omega}^t + \|u_2\|_{\infty, \Omega}^t)$$

and

$$(4.8) \quad \|\dot{v}(u_1, \epsilon) - \dot{v}(u_2, \epsilon)\|_{2,\Omega} \leq \hat{G}_2 \|\nabla p(u_1, \epsilon) - \nabla p(u_2, \epsilon)\|_{2,\Omega}$$

where  $\hat{G}_1$  depend on  $|\Omega|$ ,  $k_0$ ,  $C$  and  $\epsilon$ . In particular as  $\epsilon \rightarrow 0$  the  $\hat{G}_1$  become unbounded.

#### 4-(a) Existence, uniqueness and boundedness for $u$

The available existence and uniqueness theory cannot be directly applied here since  $\dot{v}$  is not regular enough. Let us prove uniqueness first. Suppose  $u_1$  and  $u_2$  are solutions of (4.1). Then  $w = u_1 - u_2 \in \overset{\circ}{W}^{1,2}(\Omega)$  satisfies

$$(4.9) \quad \int_{\Omega} (\nabla w - \lambda \dot{v} w) \nabla n dx = 0 \quad \forall n \in \overset{\circ}{W}^{1,2}(\Omega) \cap L^{\infty}(\Omega).$$

Taking  $n = w_n \in \overset{\circ}{W}^{1,2}(\Omega)$  in (4.9), where

$$w_n(x) = \begin{cases} n, & w_n(x) > n \\ w(n), & w(x) \in [-n, n] \\ -n, & w(x) < -n \end{cases}$$

and noting that  $\operatorname{div} \dot{v} = 0$ , results in the equation

$$\int_{\Omega} |\nabla w_n|^2 dx = 0 \quad n = 1, 2, \dots$$

Thus the convergence of  $w_n$  to  $w$  implies the equation

$$\int_{\Omega} |\nabla w|^2 dx = 0$$

and the uniqueness of  $u$ .

Existence is proved by considering a sequence of problems

$$(4.10) \quad u_n \in \overset{\circ}{W}^{1,2}(\Omega) : \int_{\Omega} (\nabla u_n - \lambda \dot{v}_n u_n) \nabla n dx = \int_{\Omega} (f_0 n - f_1 n x_1) dx \quad \forall n \in \overset{\circ}{W}^{1,2}(\Omega)$$

where  $\Phi$  is an extension of  $\phi$  into  $\Omega$  such that

$$\|\Phi\|_{\infty, \Omega} + \|\Phi\|_{W^{1,2}(\Omega)} < \Phi_0$$

for a given fixed constant  $\Phi_0$ , and  $\{\hat{v}_n\}$  is a sequence in  $D_0(\Omega)$  which converges to  $\hat{v}$  in  $J_0(\Omega)$  such that

$$\|\hat{v}_n\|_{2,\Omega} \leq \|\hat{v}\|_{2,\Omega} \equiv v.$$

Since  $\operatorname{div} \hat{v}_n = 0$ , Poincare's inequality implies that

$$((w, n)) = \int_{\Omega} (\nabla w - \lambda \hat{v}_n w) \nabla n \, dx$$

is an inner product on  $\overset{\circ}{W}{}^{1,2}(\Omega)$  which generates an equivalent topology in  $\overset{\circ}{W}{}^{1,2}(\Omega)$ .

Similarly the right hand side of (4.10) can be written as  $f(n)$  where  $f(\cdot)$  is a continuous linear functional on  $\overset{\circ}{W}{}^{1,2}(\Omega)$  and, since  $\Phi \in W^{1,2}(\Omega)$ , a standard application of the Riesz representation theorem yields the existence and uniqueness of a solution to (4.10).

We now obtain some estimates for  $u_n$  which are independent of  $n$ . Taking  $n = u_n - \Phi$  in (4.10) we obtain

$$(4.11) \quad \begin{aligned} & \int_{\Omega} \{ |\nabla(u_n - \Phi)|^2 - \frac{1}{2} \lambda \hat{v}_n \cdot \nabla(u_n - \Phi)^2 \} dx \\ &= \int_{\Omega} \{ \lambda \hat{v}_n \cdot \nabla(u_n - \Phi) \Phi - \nabla \Phi \cdot \nabla(u_n - \Phi) \} dx + \int_{\Omega} \{ f_0(u_n - \Phi) + f_1 \frac{\partial}{\partial x_1} (u_n - \Phi) \} dx. \end{aligned}$$

Observing that

$$\int_{\Omega} \hat{v}_n \cdot \nabla(u_n - \Phi)^2 dx = 0,$$

a repeated application of the inequality  $ab \leq \alpha a^2 + b^2/\alpha$ , yields the existence of a constant  $\gamma$  depending only on  $\lambda$  such that

$$(4.12) \quad \|\nabla(u_n - \phi)\|_{2,\Omega}^2 \leq \gamma(\|\phi\|_{\infty,\Omega} \|\hat{v}_n\|_{2,\Omega}^2 + \|\nabla\phi\|_{2,\Omega}^2 \sum_{i=1}^2 \|f_i\|_{2,\Omega}^2) \\ + \gamma\sigma^{-1} \|f_0\|_{2,\Omega}^2 + \sigma \|u_n - \phi\|_{2,\Omega}^2.$$

Poincaré's inequality and (4.12) imply the estimate

$$(4.13) \quad \|u_n\|_{2,\Omega}^2 + \|\nabla u_n\|_{2,\Omega}^2 \leq \gamma_0(1 + v^2),$$

where  $\gamma_0$  depends on the data  $\{\phi, f_0, f_1, f_2\}$  and  $\Omega$ .

An  $L^\infty$  bound for  $u_n$  will now be derived. Let  $\lambda > \|\phi\|_{\infty,\partial\Omega}$ . Then the function  $v = (u_n - \lambda)^+$  is in  $W^{1,2}(\Omega)$  and can be used as a test function in (4.10) to give

$$(4.14) \quad \int_{\Omega} |\nabla(u_n - \lambda)^+|^2 + \lambda \hat{v}_n \cdot \nabla u_n (u_n - \lambda)^+ dx = \int_{\Omega} f_0 (u_n - \lambda)^+ - f_1 \frac{\partial}{\partial x_1} (u_n - \lambda)^+ dx.$$

Setting

$$A_\lambda = \{x \in \Omega | u(x) > \lambda\}$$

and noting that

$$\int_{\Omega} \hat{v}_n \cdot \nabla u_n (u_n - \lambda)^+ dx = \frac{1}{2} \int_{\Omega} \hat{v}_n \cdot \nabla (u_n - \lambda)^+ dx$$

we obtain from (4.14) the inequality

$$\int_{\Omega} |\nabla(u_n - \lambda)^+|^2 dx \leq \|u_n - \lambda\|_{2,n}^2 + \sum_{i=0}^2 \int_{A_\lambda} f_i^2 dx.$$

Hölder's inequality and  $[A_\lambda]$  imply that

$$(4.15) \quad \|\nabla(u_n - \lambda)^+\|_{2,\Omega}^2 \leq \|u_n - \lambda\|_{2,n}^2 + \bar{\gamma}_1 |A_\lambda|^{1-2/2+\kappa},$$

where  $\bar{\gamma}_1$  depends only upon the data. Let  $M$  be a number larger than  $2\|\phi\|_{\infty,\partial\Omega}$  and consider the sequence of increasing levels

$$\lambda_m = M + M(1 - 2^{-m}) \quad m = 0, 1, 2, \dots.$$

A bound on  $\text{ess sup}_{x \in \Omega} u(x)$  will be obtained by taking  $\lambda = \lambda_m$  in (4.15). Hölder's inequality and Sobolev embedding (see for example inequality (2.12) on p. 45 of [10]) imply

that for  $2 < p < \infty$

$$\begin{aligned} \int_{\Omega} (u_n - t_{m+1})^{+2} dx &\leq \left\{ \int_{\Omega} (u_n - t_m)^{+p} dx \right\}^{2/p} |A_{t_{m+1}}|^{1-2/p} \\ &\leq c |\Omega|^{1/p} \|v(u_n - t_{m+1})^{+}\|_{L^2(\Omega)}^2 |A_{t_{m+1}}|^{1-2/p} \end{aligned}$$

where  $c$  depends only on  $p$ . We deduce from (4.15) that

$$(4.16) \quad \int_{\Omega} (u_n - t_{m+1})^{+2} dx \leq \bar{\gamma}_2 |A_{t_{m+1}}|^{1-2/p} \{ \|v(u_n - t_{m+1})^{+}\|_{L^2(\Omega)}^2 + |A_{t_{m+1}}|^{1-2/2+\kappa} \} .$$

Next observe that since  $t_{m+1} = t_m + M/2^{m+1}$ ,

$$\|v(u_n - t_{m+1})^{+}\|_{L^2(\Omega)}^2 \leq \|v(u_n - t_m)^{+}\|_{L^2(\Omega)}^2$$

and

$$\|v(u_n - t_m)^{+}\|_{L^2(\Omega)}^2 \geq \int_{A_{t_{m+1}}} (u_n - t_m)^{+2} dx \geq \left( \frac{M}{2^{m+1}} \right)^2 |A_{t_{m+1}}| .$$

Choose

$$p = 4(2 + \kappa)/\kappa \in (2, \infty)$$

and set

$$\sigma = \kappa/2(2 + \kappa) \in (0, 1) .$$

Substituting into (4.16), we obtain, for a constant  $\bar{\gamma}_3$  depending only upon the data

$$(4.17) \quad \int_{\Omega} (u_n - t_{m+1})^{+2} dx \leq \bar{\gamma}_3 \left[ \left( \frac{2^{m+1}}{M} \right)^{2\sigma} + \left( \frac{2^{m+1}}{M} \right)^{2+2\sigma} \right] \left( \int_{\Omega} (u_n - t_m)^{+2} dx \right)^{1+\sigma} .$$

Setting

$$y_m = \frac{1}{M^2} \int_{\Omega} (u_n - t_m)^{+2} dx$$

(4.17) implies

$$(4.18) \quad y_{m+1} \leq \bar{\gamma}_4 b^m y_m^{1+\sigma}, \quad b = 2^{2(1+\sigma)}$$

where  $\bar{Y}_4$  depends only upon the data and in particular is independent of  $n$ . Inequality (4.18) implies, with the aid of lemma 4.7 of [10], p. 66, that  $y_n \rightarrow 0$  as  $n \rightarrow \infty$  if

$$y_0 < (\bar{Y}_4)^{-1/\sigma} b^{-1/\sigma^2}.$$

Consequently if  $M$  is chosen to satisfy

$$M^2 > \bar{Y}_4^{1/\sigma} b^{1/\sigma^2} \|u_n\|_{2,\Omega}^2$$

we have that  $y_n = 0$  i.e.  $\int_{\Omega} (u_n - 2M)^+ dx = 0$  and

$$\text{ess sup}_{\Omega} u(x) < \bar{Y}_5 \max\{\|\phi\|_{\infty, \partial\Omega}, \|u_n\|_{2,\Omega}\},$$

where  $\bar{Y}_5$  depends only upon the data and is independent of  $\hat{v}$ . A bound for  $\text{ess inf}_{\Omega} u(x)$  is obtained in a similar fashion and we deduce that

$$(4.19) \quad \|u_n\|_{\infty, \Omega} < \bar{Y}_6 \max\{\|\phi\|_{\infty, \partial\Omega}, \|u_n\|_{2,\Omega}\}.$$

Passing to the limit in (4.13) and (4.19) as  $n$  tends to infinity we obtain the existence of a solution to (4.1) satisfying (4.2) and (4.3).

#### 4-(b) Hölder continuity of $u$

The results of [10] imply that the Hölder exponent and constant of  $u$  may be estimated in terms of the  $L^q(\Omega)$ ,  $q > 2$ , norm of  $\hat{v}$  and the data. However the proof of the theorem requires that these estimates are independent of  $\epsilon$  and it is known only that  $\|\hat{v}\|_{2,\Omega} < V$  where  $V$  does not depend on  $\epsilon$ , (see Section 4-(c)). In the following, the arguments of [10, p. 81, 90] are adapted to demonstrate that the extra information of  $\text{div } \hat{v} = 0$  and the space dimension being two is sufficient to conclude that  $u$  is Hölder continuous in  $\Omega$  with estimates depending only on  $\|\hat{v}\|_{2,\Omega}$  and the data.

Let  $x_0$  be an arbitrary point of  $\Omega$  and let  $B(\rho)$  denote the ball  $B(\rho) = \{x : |x - x_0| < \rho\}$ . For  $x_0 \in \Omega$ ,  $\rho$  will always be chosen so small that  $B(2\rho) \subset \Omega$ . For  $\sigma \in (0, 1)$ ,  $\zeta(\cdot)$  will always denote a smooth cutoff function such that

(i)  $\text{supp } \zeta \subset B(\mathbf{v})$

(ii)  $\zeta \equiv 1$  on  $B(p - \delta p)$  and  $\zeta(x) \in [0, 1]$

(iii)  $|\nabla \zeta| < (\delta p)^{-1}$ .

The Hölder continuity of  $u$  is shown by the judicious use of an integral inequality which follows directly from the integral identity (4.1). For arbitrary  $k \in \mathbb{R}$ , the results of 4-(a) imply that the functions

$$(4.20) \quad n = \pm(u - k)^{\frac{1}{2}}\zeta^2 \in W^{1,2}(\Omega)$$

are admissible test functions in (4.1) and setting

$$\begin{aligned} A_{k,p}^+ &\equiv \{x \in B(p) | u(x) > k\}, \\ A_{k,p}^- &\equiv \{x \in B(p) | u(x) < k\}, \end{aligned}$$

gives

$$(4.21) \quad \begin{aligned} \int_{A_{k,p}^+} |\nabla u|^2 \zeta^2 dx &= \int_{A_{k,p}^+} (\varepsilon_0(u - k)\zeta^2 - \varepsilon_1 u_{x_1} \zeta^2 - \varepsilon_1 \zeta_{x_1} 2\zeta(u - k)) dx \\ &\quad - \lambda \int_{A_{k,p}^+} \hat{v} \cdot ((u - k)\zeta^2) u dx - \int_{A_{k,p}^+} (u - k) \nabla u \cdot \nabla \zeta^2 dx. \end{aligned}$$

The inequality  $ab \leq \delta^{-1}a^2 + \delta b^2$  and the facts that  $0 < \zeta < 1$  and  $\text{div } \hat{v} = 0$  imply that the right hand side of (4.21) is bounded by

$$\begin{aligned} &\frac{1}{2} \int_{A_{k,p}^+} (\varepsilon_0^2 + (\frac{2}{\delta} + 1) \sum_{i=1}^2 \varepsilon_i^2) dx + 2\delta \int_{A_{k,p}^+} |\nabla u|^2 \zeta^2 dx \\ &\quad + \frac{1}{2} \int_{A_{k,p}^+} (u - k)^2 (1 + (2 + \frac{2}{\delta}) |\nabla \zeta|^2) dx + \lambda \int_{A_{k,p}^+} |\hat{v}| |\nabla \zeta| (u - k)^2 dx. \end{aligned}$$

Choosing  $\delta = 1/2$  and observing that

$$\int_{A_{k,p}^+} \zeta^2 dx \leq \frac{1}{2} \frac{\varepsilon_0^2}{2+\varepsilon_0} |A_{k,p}^+|^{1-\frac{2}{2+\varepsilon_0}},$$

from (4.21) we deduce the inequality

$$(4.22) \quad \|\nabla(u - k)\|_{2,B(\rho-\sigma\rho)}^2 \leq \int_{A_{k,\rho}^+} |\nabla u|^2 \zeta^2 dx \leq \gamma_2(\sigma\rho)^{-2} \|u - k\|_{2,A_{k,\rho}^+}^2$$

$$+ \gamma_2 |A_{k,\rho}^+|^{1-\frac{2}{2+\kappa}} + \gamma_2(\sigma\rho)^{-1} \int_{A_{k,\rho}^+} |\hat{v}|(u - k)^2 dx .$$

where  $\gamma_2$  is a constant depending only upon the data and is independent of  $\hat{v}$ , and  $(\sigma\rho)$  is assumed to be less than or equal to one.

Proposition 3

There exists a positive number  $\bar{s}$  depending upon the data and the  $[L^2(\Omega)]^2$  norm of  $\hat{v}$  such that either

$$\text{ess osc } u \leq 2^{-\bar{s}\kappa/2+\kappa} \frac{\text{ess osc } u}{B(R/2)}$$

or

$$\text{ess osc } u \leq (1 - 2^{-(\bar{s}+1)}) \frac{\text{ess osc } u}{B(2R)} .$$

This proposition is proved via a series of lemmas which make use of the inequality

(4.22). Setting

$$\mu = \text{ess sup}_{B(2R)} u, \quad v = \text{ess inf}_{B(2R)} u, \quad w = \text{ess osc}_{B(2R)} u \in \mu - v$$

we have, obviously, either

$$(4.23) \quad |\{x \in B(R) | u(x) > \mu - \frac{w}{2}\}| < \frac{1}{2} |B(R)|$$

or

$$(4.24) \quad |\{x \in B(R) | u(x) < \mu - \frac{w}{2}\}| < \frac{1}{2} |B(R)| .$$

It is assumed that (4.23) holds with the arguments being similar if (4.24) holds.

Setting,  $s \in \mathbb{N}$ ,

$$A_s = A_{\mu - \frac{w}{2}, R}^+$$

(4.23) implies that

$$(4.25) \quad |A_s| < \frac{1}{2} \pi R^2 \quad \forall s \in \mathbb{N}.$$

The following lemma due to De Giorgi [5] will be used.

Lemma 3

There exists a universal constant  $\beta$  such that for every pair of numbers  $\ell, k$  such that  $\ell > k$ ,

$$\begin{aligned} & (\ell - k) |\{x \in B(R) | u(x) > \ell\}|^{1/2} \\ & \leq \frac{\beta R^2}{|\{x \in B(R) | u(x) < k\}|} \int_{B(R) \cap \{u > k\} \setminus \{u > \ell\}} |\nabla u| dx \end{aligned}$$

Lemma 4

For every  $\theta_0 \in (0,1)$  there exists an integer  $s_0 = s_0(\bar{v}, \theta_0) > 0$  such that either

$$w < 2^{-s_0} R^{k/2+k}$$

or

$$|A_{s_0}| < \theta_0 \pi R^2.$$

Proof.

Applying Lemma 3 with  $\ell = \mu - w 2^{-s-1}$  and  $k = \mu - w 2^{-s}$ ,  $s > 1$  yields,

$$(4.26) \quad \left( \frac{w}{2^{s+1}} \right) |A_{s+1}|^{1/2} < \frac{\beta R^2}{|B(R) \setminus A_s|} \int_{A_s \setminus A_{s+1}} |\nabla u| dx.$$

By virtue of (4.25)  $|B(R) \setminus A_s| > \pi R^2/2$ , hence an application of Hölder's inequality to the right hand side of (4.26) results in

$$(4.27) \quad \left( \frac{w}{2^{s+1}} \right)^2 |A_{s+1}| < \left( \frac{2\beta}{\pi} \right)^2 |A_s \setminus A_{s+1}| \int_{A_s \setminus A_{s+1}} |\nabla u|^2 dx.$$

Now the inequality

$$(4.28) \quad \int_{A_s \setminus A_{s+1}} |\nabla u|^2 dx \leq \int_{A_s} |\nabla u|^2 dx = \| \nabla(u - \mu + \frac{w}{2^s})^+ \|_{2, B(R)}^2$$

holds and this last term can be estimated by making use of (4.22) written for

$k = \mu - \omega 2^{-s}$ ,  $\rho = 2R$  and  $\sigma = 1/2$  which leads to

$$\begin{aligned} \|\nabla(u - \mu + \frac{\omega}{2^s})^+\|_{2,B(R)}^2 &\leq \gamma_2 R^{-2} \|u - \mu + \frac{\omega}{2^s}\|^+_{B(2R)}^2 \\ &+ \gamma_2 |\Lambda_{\mu - \frac{\omega}{2^s}, 2R}^+|^{1-\kappa/2+\kappa} + \gamma_2 R^{-1} \int_{B(2R)} |\dot{v}| (u - \mu + \frac{\omega}{2^s})^{+2} dx \\ &\leq 4\pi (\frac{\omega}{2^s})^2 + \gamma_2 (4R^2\pi)^{1-2/2+\kappa} + \gamma_2 R^{-1} (\frac{\omega}{2^s})^2 \int_{B(2R)} |\dot{v}| dx \end{aligned}$$

since  $(u - \mu + \omega 2^{-s})^+ \leq \omega 2^{-s}$  on  $B(2R)$ . By Hölder's inequality

$$R^{-1} \int_{B(2R)} |\dot{v}| dx \leq R^{-1} \|\dot{v}\|_{2,\Omega} (\pi 4R^2)^{1/2},$$

so that

$$\|\nabla(u - \mu + \frac{\omega}{2^s})^+\|_{2,B(R)}^2 \leq \gamma_3 (1+v) (\frac{\omega}{2^s})^2 + \gamma_4 R^{\frac{2\kappa}{2+\kappa}}$$

which leads to, upon substitution into (4.27) and (4.28), the inequality

$$(4.29) \quad (\frac{\omega}{2^{s+1}})^2 |\Lambda_{s+1}| \leq (\frac{2\pi}{\kappa})^2 \{ \gamma_3 (1+v) (\frac{\omega}{2^s})^2 + \gamma_4 R^{\frac{2\kappa}{2+\kappa}} \} |\Lambda_s \setminus \Lambda_{s+1}|$$

If  $R^{\kappa/2+\kappa} < \omega 2^{-s_0}$  and  $s_0 > s$  then (4.29) implies that

$$(4.30) \quad |\Lambda_{s+1}| \leq \gamma_5 (1+v) |\Lambda_s \setminus \Lambda_{s+1}|, \quad s = 1, 2, \dots, s_0 - 1.$$

Adding these inequalities, recalling that  $|\Lambda_s| > |\Lambda_{s_0}|$  and  $|\Lambda_s| \leq \pi R^2/2$ , we obtain

$$(s_0 - 1) |\Lambda_{s_0}| \leq \gamma_5 (1+v) \pi R^2/2.$$

Hence given  $\theta_0 \in (0, 1)$  to prove the lemma let

$$(4.31) \quad s_0 = 2 - [\gamma_5 \frac{(1+v)}{2\theta_0}]$$

where  $[r]$  denotes the largest integer contained less or equal to  $r$ .

Lemma 5

Let  $H = \text{ess sup}_{B(R)} (u - u + \frac{\omega}{s_0})^+$  for any  $s_0 \in \mathbb{N}$ . There exists a number  $\theta_0 \in (0, 1)$  such that if

$$|\Lambda_{u - \frac{\omega}{s_0}, R}^+| < \theta_0 \pi R^2$$

then either

$$H < \frac{1}{2} R^{k/2+\kappa}$$

or

$$|\Lambda_{u - \frac{\omega}{s_0} + \frac{1}{2} H, R/2}^+| = 0.$$

Proof.

Consider the sequence of balls  $B(R_n)$  where

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}} \quad n = 0, 1, 2, \dots$$

and the sequence of increasing levels

$$k_n = u - \frac{\omega}{s_0} + \frac{H}{2} - \frac{H}{2^n} \quad n = 0, 1, 2, \dots$$

The inequality (4.22) can now be applied with  $k = k_n$ ,  $\rho = R_n$  and  $\sigma R_n = R/2^{n+2}$ . This leads to the inequality

$$\begin{aligned} |\nabla(u - k_n)^+|_{2, B(R_{n+1})}^2 &\leq \gamma_2 \left(\frac{2^{n+2}}{R}\right)^2 \int_{A_{k_n, R_n}}^+ (u - k_n)^2 dx \\ &+ \gamma_2 |\Lambda_{k_n, R_n}^+|^{1-\frac{2}{2+\kappa}} + \gamma_2 \left(\frac{2^{n+2}}{R}\right) \int_{A_{k_n, R_n}}^+ |\nabla(u - k_n)^+|^2 dx. \end{aligned}$$

Since  $\sup_{B(R)} (u - k_n) < \frac{H}{2} + \frac{H}{2^n} < H$  and

$$\int_{A_{k_n, R_n}} |\hat{v}|(u - k_n)^2 dx \leq \pi^2 \|v\|_{2,\Omega} |A_{k_n, R_n}^+|^{1/2},$$

from (4.32) we obtain the inequality

$$(4.33) \quad \begin{aligned} \|\nabla(u - k_n)^+\|_{2,B(R_{n+1})}^2 &\leq \gamma_2 2^{2n+4} \frac{\pi^2}{R^2} |A_{k_n, R_n}^+| \\ &+ \gamma_2 |A_{k_n, R_n}^+|^{1-\frac{2}{2+\kappa}} + \gamma_2 2^{n+2} \frac{\pi^2}{R} v |A_{k_n, R_n}^+|^{1/2} \\ &\leq \gamma_2 (1+v) 2^{2n+4} \left\{ \frac{\pi^2}{R^2} |A_{k_n, R_n}^+| + |A_{k_n, R_n}^+|^{1-\frac{2}{2+\kappa}} + \frac{\pi^2}{R} |A_{k_n, R_n}^+|^{1/2} \right\}. \end{aligned}$$

Applying lemma 3 with  $\ell = k_{n+1}$  and  $k = k_n$  over the ball  $B(R_{n+1})$  gives

$$(4.34) \quad \frac{\pi}{2^{n+1}} |A_{k_{n+1}, R_{n+1}}^+|^{1/2} \leq \frac{\beta R^2}{|B(R_{n+1}) \setminus A_{k_n, R_{n+1}}^+|} \int_{A_{k_n, R_{n+1}} \setminus A_{k_{n+1}, R_{n+1}}^+} |\nabla u| dx.$$

Since

$$|A_{k_n, R_{n+1}}^+| \leq |A_{k_0, R}^+| \leq \theta_0 \pi R^2 \text{ where } \theta_0 \in (0,1)$$

we have

$$|B(R_{n+1}) \setminus A_{k_n, R_{n+1}}^+| \geq \pi R_{n+1}^2 - \theta_0 \pi R^2 \geq \pi R^2 \left( \frac{1}{4} - \theta_0 \right) \geq \pi R^2 / 8$$

provided  $\theta_0$  is less or equal to  $1/8$ .

Consequently (4.34) implies that

$$\frac{\pi^2}{(2^{n+1})^2} |A_{k_{n+1}, R_{n+1}}^+| \leq \left( \frac{8\beta}{\pi} \right)^2 |A_{k_n, R_{n+1}}^+| \int_{A_{k_n, R_{n+1}} \setminus A_{k_{n+1}, R_{n+1}}^+} |\nabla u|^2 dx$$

and since

$$\int_{A_{k_n, R_{n+1}}} |\nabla u|^2 dx = \int_{B(R_{n+1})} |\nabla(u - k_n)^+|^2 dx$$

we have from (4.33) that

$$(4.35) \quad |A_{k_n, R_{n+1}}^+| < \gamma_6 2^{4n} (1 + v) \left[ |A_{k_n, R_n}^+|^2 / R^2 + |A_{k_n, R_n}^+|^{2-2/\kappa} / R^2 + |A_{k_n, R_n}^+|^{1+1/2} / R \right]$$

where  $\gamma_6$  is independent of  $v$  and  $n$ . Refining

$$\delta_n = |A_{k_n, R_n}^+| / R^2$$

(4.35) becomes

$$\delta_{n+1} < \gamma_6 2^{4n} (1 + v) \left[ \delta_n^2 + \frac{R^{2(1-2/2+\kappa)}}{2} \delta_n^{2-2/2+\kappa} + \delta_n^{1+1/2} \right].$$

Provided  $\theta_0 < 1/\pi$  we have that  $0 < \delta_n < 1$  and if

$$R > R^{\kappa/2+\kappa}/2$$

then

$$(4.36) \quad \delta_{n+1} < \gamma_7 2^{4n} (1 + v) \delta_n^{1+\tau}$$

where

$$\tau = \min\left\{\frac{1}{2}, 1 - \frac{2}{2+\kappa}\right\}.$$

On the basis of lemma 4.7 of [10, p. 66] it follows that provided

$$\delta_0 < (\gamma_7(1 + v))^{-1/\tau} (2^4)^{-1/\tau^2},$$

the sequence  $\delta_n$  converges to zero. Hence if we take

$$(4.37) \quad \theta_0 = \min\left\{\frac{1}{8}, \frac{1}{\pi} (\gamma_7(1 + v))^{-1/\tau} (2^4)^{-1/\tau^2}\right\}$$

then provided

$$R^2 \delta_0 = |A_{\mu-\frac{\omega}{2}, R}^+| < \theta_0^{\pi} R^2,$$

either

$$H < \frac{1}{2} R^{\kappa/2+\kappa}$$

or

$$\left| A_{\mu - \frac{\omega}{s_0} + \frac{1}{2} H, R/2}^+ \right| = 0.$$

Proof of the Proposition 3

Let  $\theta_0$  be chosen as in (4.37). By lemma 4 there exists an integer  $s_0(\theta_0, v)$  such that either

$$(4.38) \quad \omega < 2^{-s_0} R^{\kappa/2+\kappa}$$

or

$$(4.39) \quad |A_{s_0}| < \theta_0 \pi R^2$$

holds. To prove the proposition we need only consider the second possibility and show that it implies

$$(4.40) \quad \text{ess osc } u \leq \left(1 - 2^{-(s_0+1)}\right) \frac{\text{ess osc } u}{B(R/2)}$$

From lemma 5, (4.39) implies that either

$$(4.41) \quad H < \frac{1}{2} R^{\kappa/2+\kappa}$$

or

$$(4.42) \quad \left| A_{\mu - \frac{\omega}{s_0} + \frac{1}{2} H, R/2}^+ \right| = 0.$$

Suppose (4.41) holds. By the definition of  $H$ ,

$$\text{ess sup}_{B(R)} u \leq \mu - \frac{\omega}{s_0} + \frac{1}{2} R^{\kappa/2+\kappa} < \mu - \frac{\omega}{s_0} + \frac{\omega}{s_0+1} = \mu - \frac{\omega}{s_0+1}$$

since (4.38) does not hold. Hence

$$\text{ess osc}_{B(R/2)} u \leq \text{ess osc}_{B(R)} u \leq \omega - \frac{\omega}{s_0+1}$$

and we have that (4.40) holds. Otherwise if (4.42) holds then

$$\text{ess sup}_{B(R/2)} u \leq \mu - \frac{\omega}{s_0} + \frac{1}{2} H$$

$$\leq \mu - \frac{\omega}{s_0} + \frac{1}{2} \text{ess sup}_{B(R)} u - \frac{\mu}{2} + \frac{\omega}{s_0+1} \leq \mu - \frac{\omega}{s_0+1}$$

and again (4.40) holds. Consequently the number  $\bar{s}$  claimed by the proposition is the  $s_0$  given by (4.31) and (4.37).

#### Lemma 6

$u$  is Hölder continuous in  $\Omega$  with exponent

$$(4.43) \quad \alpha = \min\left\{\frac{\kappa}{2+\kappa}, \ln 1/\sigma\right\} \text{ where } \sigma = 1 - 2^{-(\bar{s}+1)}.$$

#### Proof.

For every  $x_0 \in \Omega$  and for all  $R < 1$  such that  $B(2R) \subset \Omega$ , set  $\omega(R) = \text{ess osc}_{B(R)} u$ . We have shown that for all such  $R$ , either  $\omega(R/4) < 2^{\bar{s}} R^{\kappa/2+\kappa}$  or  $\omega(R/4) < \delta \omega(R)$ . Then the lemma is an immediate consequence of Lemma 4.8 of [10, p. 66] and in particular

$$(4.44) \quad \text{osc}_{B(\rho)} u \leq G \left(\frac{\rho}{R}\right)^\alpha \quad 0 < \rho < R$$

where

$$(4.45) \quad G = 4^\alpha \max\{2\|u\|_{\infty, \Omega}, 2^{\bar{s}} R^{\kappa/2+\kappa}\}.$$

#### Remark

If the boundary datum  $\phi$  is Hölder continuous with exponent  $\beta$  then the demonstrated Hölder continuity of  $u$  carries over to  $\tilde{\phi}$  with a new Hölder exponent depending on  $V$  and  $B$ . The proof of this follows from the arguments of [10, p. 90-95] with the modifications indicated above.

4-(c). A priori estimates on the pressure

Let elements of the quotient space  $W^{1,2}(\Omega)/\mathbb{R}$  be denoted by  $[n]$ . The quotient norm

$$\| [n] \| = \inf_{n \in [n]} \| n \|_{W^{1,2}(\Omega)}$$

is equivalent to  $\| \nabla n \|_{2,\Omega}$  for  $n \in [n]$ . Thus for any  $u \in L^{\infty}(\Omega)$ ,

$$\int_{\Omega} k_{\varepsilon}(u) \nabla \xi \nabla n dx \quad [\xi], [n] \in W^{1,2}(\Omega)/\mathbb{R}$$

defines an inner product for which  $W^{1,2}(\Omega)/\mathbb{R}$  is a Hilbert space, and

$$\int_{\Omega} k_{\varepsilon}(u) \overset{+}{g}(u,x) \nabla n dx \quad v[n] \in W^{1,2}(\Omega)/\mathbb{R}$$

defines a continuous linear functional on  $W^{1,2}(\Omega)/\mathbb{R}$ . Hence the Lax-Milgram theorem implies that there exists a unique  $[p] \in W^{1,2}(\Omega)/\mathbb{R}$  such that

$$(4.46) \quad p \in [p], \int_{\Omega} k_{\varepsilon}(u) \{ \nabla p + \overset{+}{g}(u,x) \} \nabla n dx = 0 \quad \forall n \in [z] \in W^{1,2}(\Omega)/\mathbb{R}.$$

The pressure  $p \equiv p(u, \varepsilon)$  is determined up to an arbitrary constant by (4.46) whereas the gradient of  $p$ , and hence  $\overset{+}{v}(u, \varepsilon) = -k_{\varepsilon}(u) \{ \nabla p(u, z) + \overset{+}{g}(u, x) \}$ , is uniquely determined. It remains only to obtain the inequalities (4.5)-(4.8). Taking  $n = p$  in (4.4) yields (4.5) and hence the definition of  $\overset{+}{v}(u, \varepsilon)$  gives (4.6). Inequality (4.7) is proved by writing (4.4) for  $u_1$  and  $u_2$ , differencing and choosing  $n = p(u_1, \varepsilon) - p(u_2, \varepsilon)$ . Finally (4.8) is a consequence of (4.7) and the definition of  $\overset{+}{v}(u, \varepsilon)$ .

5. PROOF OF PROPOSITIONS 1 AND 2.

Fix  $\epsilon > 0$  and  $u \in L^{\infty}(\Omega)$ . Let  $p(u, \epsilon) \in W^{1,2}(\Omega)$  solve

$$(5.1) \quad \begin{cases} -\operatorname{div} k_{\epsilon}(u)[\nabla p(u, \epsilon) + \vec{g}(u, x)] = 0 & \text{in } D'(\Omega) \\ -k_{\epsilon}(u)[\nabla p(u, \epsilon) + \vec{g}(u, x)] \vec{N} = 0 \end{cases}$$

and  $\vec{v}(u, \epsilon)$  be uniquely defined by

$$(5.2) \quad \vec{v}(u, \epsilon) = -k_{\epsilon}(u)[\nabla p(u, \epsilon) + \vec{g}(u, x)].$$

Set  $F(u) \in W^{1,2}(\Omega)$  to be the unique solution of

$$(5.3) \quad \begin{cases} -\Delta F(u) + \lambda \vec{v}(u, \epsilon) \nabla F(u) = f_0 + \frac{\partial}{\partial x_i} f_i & \text{in } D'(\Omega) \\ F(u) = \phi & \text{on } \partial\Omega. \end{cases}$$

Obviously (5.1)-(5.3) are meant in the weak sense made precise earlier.

Lemma 7

- (i)  $F = L^{\infty}(\Omega) + L^{\infty}(\Omega)$  is well defined.
- (ii) There exists a constant  $M$  depending only upon the data in  $[A_0]-[A_3]$  such that if  $B(M)$  is the ball of radius  $M$  in  $L^{\infty}(\Omega)$ , then

$$F : B(M) \rightarrow B(M).$$

- (iii)  $F$  is completely continuous.

Proof. Lemmas 1 and 2 imply that

$$\|F(u)\|_{\infty, \Omega} \leq Y_1(1 + 2k_0[c|\Omega|^{1/2}\|u\|_{\infty, \Omega}^t + \|g_0\|_{2, \Omega}]) \equiv \tilde{Y}_1 + \tilde{Y}_2\|u\|_{\infty, \Omega}^t.$$

Let  $M$  be the solution of

$$\tilde{Y}_1 + \tilde{Y}_2 M^t = M > 0.$$

Such an  $M$  exists since  $t \in [0, 1]$ , and it can be determined a priori in terms of the data only. Therefore if  $u \in B(M)$  we have  $F(u) \in B(M)$ . This proves (ii). Statement (i) is a consequence of the uniqueness of  $\vec{v}(u, \epsilon)$  and  $F(u)$  defined by (5.2) and (5.3).

By lemma 1  $F(u)$  is Hölder continuous on compact subsets of  $\Omega$  with constants depending upon  $\|\vec{v}\|_{2, \Omega} \leq V$ . If  $u \in B(M)$  then by (4.6)

$$\|\vec{v}\|_{2, \Omega} \leq \tilde{Y}_1 + \tilde{Y}_2 M^t$$

where  $\tilde{Y}_1$  and  $\tilde{Y}_2$  depend only upon the data. Therefore the Hölder constants of  $F(u)$  can be a priori determined depending only on the data. Consequently  $F$  is compact. In order to show continuity of  $F$ , let  $\{u_j\}$  be a sequence in  $B(M)$  such that

$u_j \rightarrow u_0 \in B(M)$  uniformly in  $\Omega$ . Writing (5.3) in the weak form for  $u_j$  and  $u_0$  we obtain, by differencing, the equation

$$\begin{aligned} & \int_{\Omega} (\nabla(F(u_j) - F(u_0)) - \lambda \hat{v}(u_j, \epsilon)(F(u_j) - F(u_0))) \nabla v dx \\ &= - \int_{\Omega} (\hat{v}(u_j, \epsilon) - \hat{v}(u_0, \epsilon)) \nabla F(u_0) v dx \quad \forall v \in \overset{\circ}{W}{}^{1,2}(\Omega). \end{aligned}$$

We take  $v = F(u_j) - F(u_0)$  and observe that the second integral on the left hand side vanishes for this choice of  $v$ , since  $\operatorname{div} \hat{v} = 0$  in  $D'(\Omega)$ . Thus we obtain

$$\int_{\Omega} |\nabla(F(u_j) - F(u_0))|^2 dx \leq G(M, \|\nabla F(u_0)\|_{2,\Omega}) \|\hat{v}(u_j, \epsilon) - \hat{v}(u_0, \epsilon)\|_{2,\Omega}$$

In view of (4.7) and (4.8), the following inequality holds

$$\|\nabla(F(u_j) - F(u_0))\|_{2,\Omega}^2 \leq G(M, \|\nabla F(u_0)\|_{2,\Omega}, \text{data}, \epsilon) \cdot \{ \|u_j - u_0\|_{\infty, \Omega} + \|\hat{G}_j - \hat{G}_0\|_{\infty, \Omega} \}$$

where  $\hat{G}_j = \hat{g}(u_j, x)$ . Letting  $j \rightarrow \infty$  we see that, since  $\hat{g}(\cdot, x)$  is continuous by [A<sub>3</sub>],

$$F(u_j) \rightarrow F(u_0) \text{ strongly in } \overset{\circ}{W}{}^{1,2}(\Omega)$$

However the uniform boundedness and Hölder continuity of  $\{F(u_j)\}$  implies for every subsequence  $\{F(u'_{j_k})\}$  the existence of a subsequence  $u'_{j_k}$  for which

$$F(u'_{j_k}) \rightharpoonup \hat{F} \text{ in } L^\infty(\Omega).$$

Hence  $\hat{F} = F(u_0)$  and for the entire sequence

$$F(u_j) \rightarrow F(u_0) \text{ in } L^\infty(\Omega).$$

$F$  is continuous and the lemma is proved.

By the Leray-Schauder fixed point theorem  $F$  has a fixed point in  $L^\infty(\Omega)$ . Therefore we can conclude that  $\forall \epsilon > 0$  problem (2.2)-(2.5) has a solution and Propositions 1 and 2 then follow.

Remark

In the case  $t = 1$ , there exists an  $M > 0$  solving (5.4) when  
 $\tilde{\gamma}_2 = \gamma_1 2k_0 C |\Omega|^{1/2} < 1$ . Thus we have existence for  $t = 1$  when the data is sufficiently small.

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